

Lecture about finding $|\pm_x\rangle$ and then $|\pm_y\rangle$. There are two conventional choices to make: relative phase for ${}_x\langle+|- \rangle_x$ and ${}_y\langle+|+ \rangle_x$.

So far, we've talked about how to calculate measurement probabilities if you know the input and output quantum states using the probability postulate:

$$\mathcal{P} = |\langle\psi_{out}|\psi_{in}\rangle|^2$$

Now we're going to do this process in reverse.

I want to be able to relate the output states of Stern-Gerlach analyzers oriented in different directions to each other (like $|\pm_x\rangle$ and $|\pm_y\rangle$ to $|\pm_z\rangle$). Since $|\pm_z\rangle$ forms a basis, I can write any state for a spin-1/2 system as a linear combination of those states, including these special states.

I'll start with $|+\rangle_x$ written in the S_z basis with general coefficients:

$$|+\rangle_x = a|+\rangle + be^{i\phi}|-\rangle$$

Notice that:

(1) a , b , and ϕ are all real numbers; (2) the relative phase is loaded onto the second coefficient only. My job is to use measurement probabilities to determine a , b , and ϕ .

I'll prepare a state $|+\rangle_x$ and then send it through x , y , and z analyzers. When I do that, I see the following probabilities:

Input = $ +\rangle_x$	S_x	S_y	S_z
$P(\hbar/2)$	1	1/2	1/2
$P(-\hbar/2)$	0	1/2	1/2

First, looking at the probability for the S_z components:

$$(S_z = +\hbar/2) = |\langle+|+\rangle_x|^2 = 1/2$$

Plugging in the $|+\rangle_x$ written in the S_z basis:

$$1/2 = \left| \langle+| \left(a|+\rangle + be^{i\phi}|-\rangle \right) \right|^2$$

Distributing the $\langle+|$ through the parentheses and use orthonormality:

$$\begin{aligned} 1/2 &= \left| a\langle+|+\rangle + be^{i\phi}\langle+|-\rangle \right|^2 \\ &= |a|^2 \\ \rightarrow a &= \frac{1}{\sqrt{2}} \end{aligned}$$

Similarly, looking at $S_z = -\hbar/2$:

$$\begin{aligned}
 (S_z = +\hbar/2) &= |\langle -|+\rangle_x|^2 = 1/2 \\
 1/2 &= \left| \langle -| \left(a|+\rangle + be^{i\phi}|-\rangle \right) \right|^2 \\
 1/2 &= \left| a \langle -|+\rangle + be^{i\phi} \langle -|-\rangle \right|^2 \\
 &= |be^{i\phi}|^2 \\
 &= |b|^2 (e^{i\phi})(e^{-i\phi}) \\
 &\rightarrow b = \frac{1}{\sqrt{2}}
 \end{aligned}$$

I can't yet solve for ϕ but I can do similar calculations for $|- \rangle_x$:

Input = $ - \rangle_x$	S_x	S_y	S_z
$P(\hbar/2)$	0	1/2	1/2
$P(-\hbar/2)$	1	1/2	1/2

$$\begin{aligned}
 |- \rangle_x &= c|+\rangle + de^{i\gamma}|-\rangle \\
 (S_z = +\hbar/2) &= |\langle +|- \rangle_x|^2 = 1/2 \\
 \rightarrow c &= \frac{1}{\sqrt{2}} \\
 (S_z = +\hbar/2) &= |\langle -|- \rangle_x|^2 = 1/2 \\
 \rightarrow d &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

So now I have:

$$\begin{aligned}
 |+\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\beta}|-\rangle \\
 |- \rangle_x &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\gamma}|-\rangle
 \end{aligned}$$

I know $\beta \neq \gamma$ because these are not the same state - they are orthogonal to each other:

$$\begin{aligned}
 0 &= {}_x \langle +|- \rangle_x \\
 &= \left(\frac{1}{\sqrt{2}} \langle +| + \frac{1}{\sqrt{2}} e^{i\beta} \langle -| \right) \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\gamma} |-\rangle \right)
 \end{aligned}$$

Now FOIL like mad and use orthonormality:

$$\begin{aligned}
 0 &= \frac{1}{2} \left(\langle + | + \rangle + e^{i\gamma} \langle + | - \rangle + e^{i\beta} \langle - | + \rangle + e^{i(\gamma-\beta)} \langle - | - \rangle \right) \\
 &= \frac{1}{2} (1 + e^{i(\gamma-\beta)}) \\
 \rightarrow e^{i(\gamma-\beta)} &= -1
 \end{aligned}$$

This means that $\gamma - \beta = \pi$. I don't have enough information to solve for β and γ , but there is a one-time conventional choice made that $\beta = 0$ and $\gamma = 1$, so that:

$$\begin{aligned}
 |+\rangle_x &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\theta} |-\rangle \\
 |-\rangle_x &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\pi} |-\rangle \\
 \rightarrow |+\rangle_x &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \\
 |-\rangle_x &= \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle
 \end{aligned}$$

When $|\pm\rangle_y$ is the input state:

Input = $ +\rangle_y$	S_x	S_y	S_z
$P(\hbar/2)$	1/2	1	1/2
$P(-\hbar/2)$	1/2	0	1/2

Input = $ -\rangle_y$	S_x	S_y	S_z
$P(\hbar/2)$	1/2	0	1/2
$P(-\hbar/2)$	1/2	1	1/2

The calculations proceed in the same way. The S_z probabilities give me:

$$\begin{aligned}
 |+\rangle_y &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\alpha} |-\rangle \\
 |-\rangle_y &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\theta} |-\rangle
 \end{aligned}$$

The orthogonality between $|+\rangle_y$ and $|-\rangle_y$ mean that $\theta - \alpha = \pi$.

But I also know the S_x probabilities and how to write $|ket\pm_x\rangle$ in the S_z basis. For an input of $|+\rangle_y$:

$$\begin{aligned}
 (S_x = +\hbar/2) &= |{}_x\langle +|+_y\rangle|^2 = 1/2 \\
 1/2 &= \left| \left(\frac{1}{\sqrt{2}} \langle +| + \frac{1}{\sqrt{2}} \langle -| \right) \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\alpha} |-\rangle \right) \right|^2 \\
 1/2 &= \left| \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \langle +|+\rangle + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} e^{i\alpha} \langle -|-\rangle \right|^2 \\
 &= \frac{1}{4} |1 + e^{i\alpha}|^2 \\
 &= \frac{1}{4} (1 + e^{i\alpha})(1 + e^{-i\alpha}) \\
 &= \frac{1}{4} (2 + e^{i\alpha} + e^{-i\alpha}) \\
 &= \frac{1}{4} (2 + 2\cos\alpha) \\
 \frac{1}{2} &= \frac{1}{2} + \frac{1}{2} \cos\alpha \\
 0 &= \cos\alpha \\
 \rightarrow \alpha &= \pm \frac{\pi}{2}
 \end{aligned}$$

Here, again, I can't solve exactly for alpha (or θ), but the convention is to choose $\alpha = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, making

$$\begin{aligned}
 |+\rangle_y &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\pi/2} |-\rangle \\
 |-\rangle_y &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i3\pi/2} |-\rangle \\
 \rightarrow |+\rangle_y &= \frac{1}{\sqrt{2}} |+\rangle + \frac{i}{\sqrt{2}} |-\rangle \\
 |-\rangle_y &= \frac{1}{\sqrt{2}} |+\rangle - \frac{i}{\sqrt{2}} |-\rangle
 \end{aligned}$$

If I use these two conventions for the relative phases, then I can write down $|\pm\rangle_n$ in an arbitrary direction described by the spherical coordinates θ and ϕ as:

Discuss the generalize eigenstates:

$$\begin{aligned}
 |+\rangle_n &= \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} e^{i\phi} |-\rangle \\
 |-\rangle_n &= \sin \frac{\theta}{2} |+\rangle - \cos \frac{\theta}{2} e^{i\phi} |-\rangle
 \end{aligned}$$

And how the $|\pm\rangle_x$ and $|\pm\rangle_y$ are consistent.