1 Free energy of a harmonic oscillator

A one-dimensional harmonic oscillator has an infinite series of equally spaced energy states, with $\varepsilon_n = n\hbar\omega$, where n is an integer ≥ 0, and ω is the classical frequency of the oscillator. We have chosen the zero of energy at the state $n = 0$ *which we can get away with here, but is not actually the zero of energy!* To find the true energy we would have to add a $\frac{1}{2}\hbar\omega$ for each oscillator.

(a) Show that for a harmonic oscillator the free energy is

$$F = k_B T \log\left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right) \tag{1}$$

Note that at high temperatures such that $k_B T \gg \hbar\omega$ we may expand the argument of the logarithm to obtain $F \approx k_B T \log\left(\frac{\hbar\omega}{kT}\right)$.

(b) From the free energy above, show that the entropy is

$$\frac{S}{k_B} = \frac{\frac{\hbar\omega}{kT}}{e^{\frac{\hbar\omega}{kT}} - 1} - \log\left(1 - e^{-\frac{\hbar\omega}{kT}}\right) \tag{2}$$

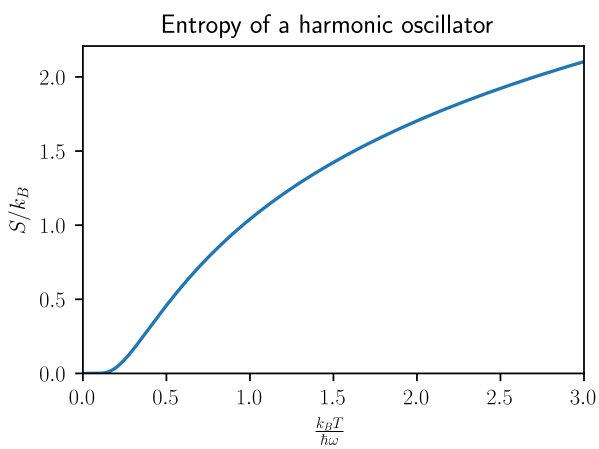

Figure 1: Entropy of a simple harmonic oscillator

This entropy is shown in the nearby figure, as well as the heat capacity.

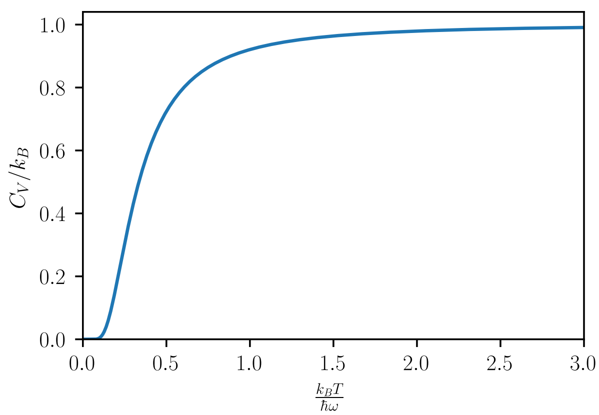

Figure 2: Heat capacity of a simple harmonic oscillator