1 Radiation in an empty box

As discussed in class, we can consider a black body as a large box with a small hole in it. If we treat the large box a metal cube with side length L and metal walls, the frequency of each normal mode will be given by:

$$\omega_{n_x n_y n_z} = \frac{\pi c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} \tag{1}$$

where each of n_x, n_y, and n_z will have positive integer values. This simply comes from the fact that a half wavelength must fit in the box. There is an additional quantum number for polarization, which has two possible values, but does not affect the frequency. **Note that in this problem I'm using different boundary conditions from what I use in class. It is worth learning to work with either set of quantum numbers.** Each normal mode is a harmonic oscillator, with energy eigenstates $E_n = n\hbar\omega$ where we will not include the zero-point energy $\frac{1}{2}\hbar\omega$, since that energy cannot be extracted from the box. (See the Casimir effect for an example where the zero point energy of photon modes does have an effect.)

Note This is a slight approximation, as the boundary conditions for light are a bit more complicated. However, for large n values this gives the correct result.

(a) Show that the free energy is given by

$$F = 8\pi \frac{V(kT)^4}{h^3c^3}\int_0^\infty \ln\left(1 - e^{-\xi}\right)\xi^2 d\xi \tag{2}$$

$$= -\frac{8\pi^5}{45}\frac{V(kT)^4}{h^3c^3} \tag{3}$$

$$= -\frac{\pi^2}{45}\frac{V(kT)^4}{\hbar^3c^3} \tag{4}$$

provided the box is big enough that $\frac{\hbar c}{LkT} \ll 1$. Note that you may end up with a slightly different dimensionless integral that numerically evaluates to the same result, which would be fine. I also do not expect you to solve this definite integral analytically, a numerical confirmation is fine. **However, you must manipulate your integral until it is dimensionless and has all the dimensionful quantities removed from it!**

(b) Show that the entropy of this box full of photons at temperature T is

$$S = \frac{32\pi^5}{45}kV\left(\frac{kT}{hc}\right)^3 \tag{5}$$

$$= \frac{4\pi^2}{45}kV\left(\frac{kT}{\hbar c}\right)^3 \tag{6}$$

(c) Show that the internal energy of this box full of photons at temperature T is

$$\frac{U}{V} = \frac{8\pi^5}{15}\frac{(kT)^4}{h^3c^3} \tag{7}$$

$$= \frac{\pi^2}{15}\frac{(kT)^4}{\hbar^3c^3} \tag{8}$$