1 Energy of a relativistic Fermi gas

For electrons with an energy $\varepsilon \gg mc^2$, where m is the mass of the electron, the energy is given by $\varepsilon \approx pc$ where p is the momentum. For electrons in a cube of volume $V = L^3$ the momentum takes the same values as for a non-relativistic particle in a box.

(a) Show that in this extreme relativistic limit the Fermi energy of a gas of N electrons is given by

$$\varepsilon_F = \hbar \pi c \left(\frac{3n}{\pi} \right)^{\frac{1}{3}} \tag{1}$$

where $n \equiv \frac{N}{V}$ is the number density.

(b) Show that the total energy of the ground state of the gas is

$$U_0 = \frac{3}{4} N \varepsilon_F \tag{2}$$