An ice cream cone is to be dipped in chocolate. The cone can be described by the equation $z^2 = 9\,(x^2+y^2)$, with $0 \le z \le 9$ and x, y, and z in centimeters. The dipping process is such that the resulting (surface) density of chocolate on the cone is given by $\sigma = 1 - \frac{z}{9}$ in grams per square centimeter. Find the total amount of chocolate on the cone.

(There is no ice cream on the cone!)

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Solution The subtlety in this problem is that the slant height of the cone is *not* the same as the vertical height. Using ***dvecr*** ensures that the area element is correct!

We need to calculate $dA = |d\vec{\boldsymbol{A}}| = |d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2|$, starting from the cylindrical expression

$$d\vec{\boldsymbol{r}} = ds\,\hat{\boldsymbol{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\boldsymbol{z}}$$

It is easy to see that $d\vec{\boldsymbol{r}}_1 = s\,d\phi\,\hat{\boldsymbol{\phi}}$, since s and z are constant as you go around the cone. However, along the slanted edge of the cone, both of z and s are changing, so $d\vec{\boldsymbol{r}}_2 = ds\,\hat{\boldsymbol{s}} + dz\,\hat{\boldsymbol{z}}$. In order to express this last expression in terms of a single variable, we must relate z and s, which can be done using proportional reasoning.

Drawing similar triangles vertically, we can conclude that

$$\frac{R}{H} = \frac{s}{z}$$

where the radius $R = 3$ and height $H = 9$ are given. In other words, we have $z = 3s$, and therefore $dz = 3\,ds$.

We can use these expressions to eliminate either s or z. Choosing the latter, we obtain

$$dA = |s\,d\phi\,\hat{\boldsymbol{\phi}} \times (\hat{\boldsymbol{s}} + 3\,\hat{\boldsymbol{z}})\,ds| = |\hat{\boldsymbol{s}} + 3\,\hat{\boldsymbol{z}}|\,s\,ds\,d\phi = \sqrt{10}\,s\,ds\,d\phi$$

We can now evaluate the desired integral, namely

$$\int \sigma\,dA = \int_0^{2\pi} \int_0^3 \left(1 - \frac{s}{3}\right) \sqrt{10}\,s\,ds\,d\phi = 20\sqrt{10}\,\pi \left(\frac{s^2}{2} - \frac{s^3}{9}\right)\Bigg|_0^3$$

yielding a final answer of $3\pi\sqrt{10}$ grams.