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A fishing net S is in the shape of a triangular trough, as shown in the picture. The triangular sides are at $x = 0$ and $x = 5$, the rectangular sides are at 45° to the vertical, and the bottom is at $z = 0$; all lengths are measured in cm. There is no netting across the top, which is at $z = 1$. Water is draining out of the net; the motion of the water is described by the vector field $\vec{\boldsymbol{F}} = \rho\left(ae^{\kappa z^2}\hat{\boldsymbol{y}} - b\,\hat{\boldsymbol{z}}\right)$ where $a = 3\,\frac{\text{cm}}{\text{s}}$, $b = 5\,\frac{\text{cm}}{\text{s}}$, $\kappa = 2\,\text{cm}^{-2}$, and ρ is the (constant) density of the water in $\frac{\text{g}}{\text{cm}^3}$. The goal of this problem is to find the best way to evaluate the flux

$$\iint \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}}$$

of water *down* through S.

- Set up the above surface integral, **but do not evaluate it**
  Your answer should be ready to integrate; among other things, all substitutions should be made, and you should determine the correct limits.

- **Use the Divergence Theorem** to find another way to do the problem.
  This time, complete the computation.

Solution The fishing net has four sides, so the total flux is the sum of the flux out of each side. However, the two triangular sides have normal vector $\hat{\boldsymbol{x}}$, and the $\vec{\boldsymbol{F}}$ has no x component, so the flux out of each of these two sides is zero.

On the remaining sides, it is important to draw an explicit chopping, clearly labeling $d\vec{\boldsymbol{r}}_1$ and $d\vec{\boldsymbol{r}}_2$ in each case. Many choices are possible, but it is essential to ensure both that $d\vec{\boldsymbol{A}}$ is *downward*-pointing, and that the limits of integration match the directions of the chosen infinitesimal displacement vectors.

In the figure, the x axis points toward the front left, the y axis points to the right, and the z axis points up. We begin with the "near" side, on which $z = y$. For $d\vec{\boldsymbol{A}}$ to point *down*, we can choose $d\vec{\boldsymbol{r}}_1$ to point *up*, and $d\vec{\boldsymbol{r}}_2$ to point to the *left*. In other words,

$$d\vec{\boldsymbol{r}}_1 = dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}$$
$$d\vec{\boldsymbol{r}}_2 = dx\,\hat{\boldsymbol{x}}$$

with x, y, and z all increasing. On the "far" side, where $z = -y$, we can choose $d\vec{\boldsymbol{r}}_1$ to point along the positive x axis (so *left* in the figure), and $d\vec{\boldsymbol{r}}_2$ to point *down*, so that

$$d\vec{\boldsymbol{r}}_1 = dx\,\hat{\boldsymbol{x}}$$
$$d\vec{\boldsymbol{r}}_2 = dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}$$

where x and z are again *increasing*, but y is now *decreasing*!

We can now separately eliminate one of y and z on each side; we choose to eliminate y. Thus, on the near side

$$d\vec{\boldsymbol{A}} = dz\,(\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) \times dx\,\hat{\boldsymbol{x}} = (\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})\,dz\,dx$$

and on the far side

$$d\vec{\boldsymbol{A}} = dx\,\hat{\boldsymbol{x}} \times dz\,(-\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) = -(\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})\,dz\,dx$$

We can now compute the dot product, yielding on the near side

$$\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}} = \int_0^5 \int_0^1 \rho \left(3e^{2z^2} + 5\right)\,dz\,dx$$

and on the far side

$$\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}} = \int_0^5 \int_0^1 \rho \left(-3e^{2z^2} + 5\right)\,dz\,dx$$

These integrals can not be evaluated separately, although the problematic terms cancel if you add the integrands *before* evaluating.

Meanwhile, the Divergence Theorem says that the *total* flux *out of* a *closed* fishing net equals the total divergence inside the net. But it is easy to see that the divergence $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}$ is zero everywhere, so that the total divergence inside the net is also zero. Thus, the flux *down* through the four sides of the actual fishing net *plus* the flux *up* through the missing top must be zero. Equivalently, the flux *down* through the missing top must equal the flux *down* through the fishing net; since the divergence vanishes, what comes in must go out.

The easier way to determine the flux down through the fishing net is therefore to compute the flux down through the missing top instead. On this surface, we clearly have $d\vec{\boldsymbol{A}} = -dx\,dy\,\hat{\boldsymbol{z}}$, so

$$\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}} = \int_{-1}^1 \int_0^5 5\rho\,dx\,dy = 50\rho$$

in grams per second.