

Consider a parabola $y = \alpha x^2$ for $0 \text{ m} \leq x \leq 5 \text{ m}$. Find the total charge if the charge density on the parabola is proportional to the width of the parabola and varies from 0 Coulombs per meter at $x = 0 \text{ m}$ to 11 Coulombs per meter at $x = 5 \text{ m}$.

Solution This problem illustrates the "use what you know" strategy.

First we need to find a formula for the charge density, λ . The words "proportional to the width" mean that the charge density varies linearly with respect to x since x is a parameter that measures (half of) the width of the parabola in the xy -plane. Therefore we know that $\lambda(x) = \alpha x + \beta$. We can use the conditions that $\lambda(0 \text{ m}) = 0 \text{ C/m}$ and $\lambda(5 \text{ m}) = 11 \text{ C/m}$ to find the values of α and β .

$$0 \text{ C/m} = \lambda(0 \text{ m}) = \alpha(0 \text{ m}) + \beta \quad (1)$$

$$\Rightarrow \beta = 0 \text{ C/m} \quad (2)$$

$$11 \text{ C/m} = \lambda(5 \text{ m}) = \alpha 5 \text{ m} + 0 \text{ C/m} \quad (3)$$

$$\Rightarrow \alpha = \frac{11}{5} \text{ C/m}^2 \quad (4)$$

$$\rightarrow \lambda(x) = \left(\frac{11}{5} \text{ C/m}^2\right)x \quad (5)$$

Next, we need to find $d\ell = |d\vec{r}|$ along the parabola. Start with the expression for $d\vec{r}$ on the plane in rectangular coordinates

$$d\vec{r} = dx \hat{x} + dy \hat{y}$$

and then plug in what you know about the equation of a parabola and its differential:

$$y = 3x^2 \quad (6)$$

$$\Rightarrow dy = 6x dx \quad (7)$$

$$\Rightarrow d\vec{r} = dx \hat{x} + 6x dx \hat{y} \quad (8)$$

$$= dx(\hat{x} + 6x \hat{y}) \quad (9)$$

Now find the magnitude of $d\vec{r}$

$$d\ell = |d\vec{r}| = \sqrt{dx(\hat{x} + 6x \hat{y}) \cdot dx(\hat{x} + 6x \hat{y})} \quad (10)$$

$$= \sqrt{dx^2(1 + 36x^2)} \quad (11)$$

$$= \sqrt{1 + 36x^2} |dx| \quad (12)$$

Plug all of these results into the formula for the total charge of a one-dimensional charge distribution.

$$Q_{\text{total}} = \int \lambda d\ell \quad (13)$$

$$= \int_0^5 \frac{11}{5} x \sqrt{1 + 36x^2} |dx| \quad (14)$$

We can remove the absolute values signs from $|dx|$ because we are integrating from small values of x to large values of x so dx is always positive. The integral can be performed with a simple substitution, $u = 1 + 36x^2$.