Calculate the momentum space wave function for a particle in an energy eigenstate of the infinite square well.

Solution

$$\psi_E(p) = \langle p|E_n\rangle = \langle p| \left(\int_{-\infty}^{\infty} |x\rangle \langle x| \, dx \right) |E_n\rangle$$

$$= \int_{-\infty}^{\infty} \langle p|x\rangle \langle x|E_n\rangle \, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \, \psi_E(x) \, dx$$

The energy eigenstates are:

$$\psi_E(x) = \begin{cases} \sqrt{\frac{2}{L}} \, \sin \frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{else} \end{cases}$$

So, my transformation becomes:

$$\langle p|\psi\rangle = \phi(p) = \int_{-\infty}^{0} 0 \, dx + \int_{0}^{L} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \, \sin \frac{n\pi x}{L} \, dx + \int_{0}^{\infty} 0 \, dx$$

$$= \int_{0}^{L} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \, \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{0}^{L} e^{-ipx/\hbar} \left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) dx$$

Let $\frac{n\pi}{L} = \frac{p_0}{\hbar}$:

$$
\begin{aligned}
\langle p|\psi\rangle = \phi(p) &= \frac{1}{2i\sqrt{2\pi\hbar}} \int_0^L e^{-ipx/\hbar} \left(e^{ip_0x/\hbar} - e^{-ip_0x/\hbar}\right) dx \\
&= \frac{1}{2i\sqrt{2\pi\hbar}} \int_0^L \left(e^{-i(p-p_0)x/\hbar} - e^{-i(p+p_0)x/\hbar}\right) dx \\
&= \frac{1}{2\sqrt{2\pi\hbar}} \Big(\frac{e^{-i(p-p_0)x/\hbar}}{-i(p-p_0)/\hbar}\Big|_0^L - \frac{e^{-i(p+p_0)x/\hbar}}{-i(p+p_0)/\hbar}\Big|_0^L \Big) \\
&= \sqrt{\frac{\hbar}{8\pi}} \Big(\frac{e^{-i(p-p_0)L/\hbar} - 1}{(p-p_0)} - \frac{e^{-i(p+p_0)L/\hbar} - 1}{(p+p_0)} \Big)
\end{aligned}
$$

This is not a simple sum of momentum eigenstates.

To put this into a slightly better form, I'll use the fact that:

$$
\begin{aligned}
e^{-i\theta} - 1 &= e^{-i\theta/2}(e^{-i\theta/2} - e^{+i\theta/2}) \\
&= e^{-i\theta/2}\left(-2i\sin\tfrac{\theta}{2}\right) \\
&= -i2e^{-i\theta/2}\sin\tfrac{\theta}{2}
\end{aligned}
$$

Making the substitution:

$$
\phi(p) = \frac{-2i\hbar}{\sqrt{8\pi\hbar}} \left(\frac{e^{-i(p-p_0)L/2\hbar}\sin\frac{(p-p_0)L}{2\hbar}}{p-p_0} - \frac{e^{-i(p+p_0)L/2\hbar}\sin\frac{(p+p_0)L}{2\hbar}}{p+p_0} \right)
$$

To better understand this, I'll calculate the probability density and plot it:

$$
\begin{aligned}
|\phi(p)|^2 &= \frac{\hbar}{2\pi}\left|\frac{e^{-i(p-p_0)L/2\hbar}\sin\frac{(p-p_0)L}{2\hbar}}{p-p_0} - \frac{e^{-i(p+p_0)L/2\hbar}\sin\frac{(p+p_0)L}{2\hbar}}{p+p_0}\right|^2 \\
&= \frac{\hbar}{2\pi}\Bigg(\frac{\sin^2\frac{(p-p_0)L}{2\hbar}}{(p-p_0)^2} + \frac{\sin^2\frac{(p+p_0)L}{2\hbar}}{(p+p_0)^2} \\
&\quad - \frac{e^{-i(p-p_0)L/2\hbar}e^{+i(p+p_0)L/2\hbar}\sin\frac{(p-p_0)L}{2\hbar}\sin\frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)} \\
&\quad - \frac{e^{+i(p-p_0)L/2\hbar}e^{-i(p+p_0)L/2\hbar}\sin\frac{(p-p_0)L}{2\hbar}\sin\frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)}\Bigg) \\
&= \frac{\hbar}{2\pi}\Bigg[\frac{\sin^2\frac{(p-p_0)L}{2\hbar}}{(p-p_0)^2} + \frac{\sin^2\frac{(p+p_0)L}{2\hbar}}{(p+p_0)^2} \\
&\quad - \frac{\cos\frac{p_0L}{\hbar}\sin\frac{(p-p_0)L}{2\hbar}\sin\frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)}\Bigg]
\end{aligned}
$$

Remembering that $p_0 = \frac{n\pi\hbar}{L}$, the cosine term simplifies:

$$
\begin{aligned}
|\phi(p)|^2 = \frac{\hbar}{2\pi}\Bigg[&\frac{\sin^2\frac{(p-p_0)L}{2\hbar}}{(p-p_0)^2} + \frac{\sin^2\frac{(p+p_0)L}{2\hbar}}{(p+p_0)^2} \\
&- \cos n\pi\frac{\sin\frac{(p-p_0)L}{2\hbar}\sin\frac{(p+p_0)L}{2\hbar}}{(p-p_0)(p+p_0)}\Bigg]
\end{aligned}
$$

Plotting the probability density, (setting $\hbar = 1$, $L = 1$, and $n = 1$) I see two strong peaks at the $\pm p_0$ but not quite delta functions.

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